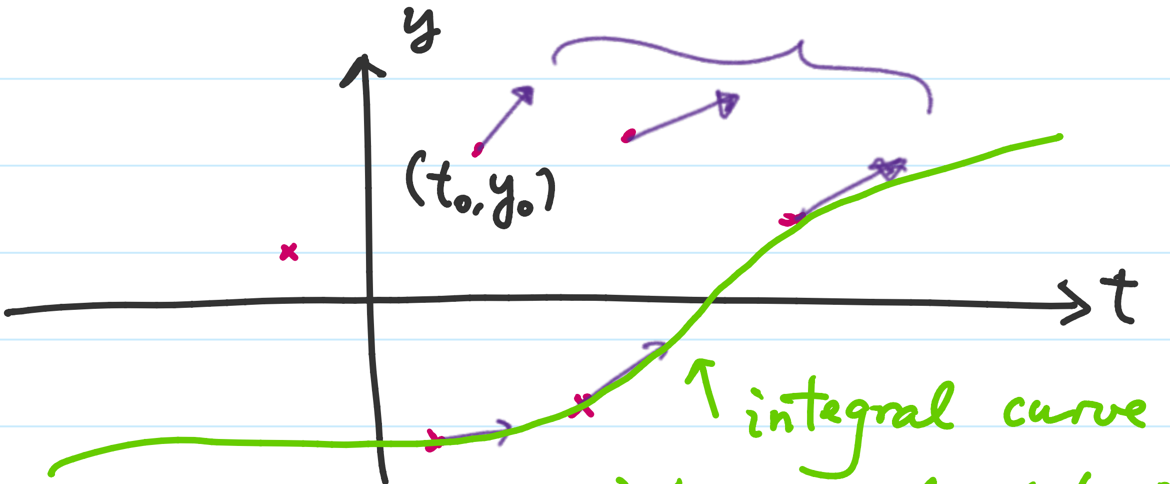
Lecture 3: 11-09-18

§ Geometric view: $y' = f(t,y)$

The vector $(1, f(t_0,y_0))$ at each point (t_0,y_0) is put together to give a vector field.



Def:
- A vector field $v(t,y)$ is an association of a vector $v(t,y) = (v_1(t,y), v_2(t,y))$ for each point (t,y).
- $\gamma(s)$ is an integral curve of v if

$$\gamma'(s) = v(\gamma(s))$$

Prop: $y(t)$ is a solution to $y' = f(t,y)$

$\iff$ its graph $\gamma(t) := (t, y(t))$ is an integral curve for the vector field $v(t,y) = (1, f(t,y))$

§ Separable equation: (for non-Linear equation).

Ex: IVP: $\begin{cases} y'(t) = \dfrac{\sin(t)}{1-y^2} \\ y(t_0) = y_0 . \end{cases}$

First rewrite: $(1-y^2)\, y' = \sin(t)$

Idea: $\left(y - \frac{1}{3}y^3\right)' = (1-y^2)\,y'$

by chain rule.

Hence: $\boxed{y - \frac{1}{3}y^3 = -\cos(t) + C .}$

Inital value:

$$y_0 - \frac{1}{3}y_0^3 = -\cos(t_0) + C .$$

$\Rightarrow$ Solution to IVP

$$y - \frac{1}{3}y^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3$$

we call it an implicit solution to the IVP.

because of non-linearity

Rk: We have to use implicit function theorem to solve for y as function of t.

Def: A first order ODE is called separable if it can be rewritten as $\boxed{M(t) + N(y)\frac{dy}{dt} = 0 .}$

Idea: suppose we have anti-derivatives

$$m'(t) = M, \qquad n'(t) = N.$$

$\Rightarrow$ we can write

$$\frac{d}{dt}\left(m(t) + n(y(t))\right) = 0$$

$\Rightarrow$ $m(t) + n(y(t)) = C$ is an implicity general solution.

- Suppose we look for IVP

then we have

$$m(t_0) + n(y_0) = C$$

$$\Rightarrow \boxed{n(y(t)) - n(y_0) = m(t_0) - m(t)}$$

Example: $y' = p(t)\,y$ (assume $y \neq 0$)

$$\Rightarrow \underbrace{-p(t)}_{M(t)} + \underbrace{\frac{1}{y}}_{N(t)}\frac{dy}{dt} = 0.$$

$$\Rightarrow \log|y(t)| - \log|y_0| = \int_t^{t_0} -p(s)\,ds.$$

$$\Rightarrow y(t) = y_0\, e^{\int_{t_0}^{t} p(s)\,ds}$$

§ Exact equation:

Recall: We have solve separable equation of the type

$$M(t) + N(y)\frac{dy}{dt} = 0.$$

find $m'(t) = M(t)$, $n'(y) = N(y)$

What if we have

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \quad ?$$

Idea: we look for function $\psi(t,y)$ s.t.

$$\boxed{\frac{\partial \psi}{\partial t} = M, \quad \frac{\partial \psi}{\partial y} = N}$$

Def: Exact equation

separable equation $\Rightarrow$

If such ψ exist:

we can write

$$\frac{\partial \psi}{\partial t}\frac{dt}{dt} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} = 0.$$

Chain rule $\Longrightarrow$ $\frac{d}{dt}(\psi(t, y(t))) = 0$

Hence we have $\psi(t, y(t)) = c$

Suppose we have the initial condition $y(t_0) = y_0$

$$\Longrightarrow \boxed{\psi(t, y(t)) \equiv \psi(t_0, y_0)}$$

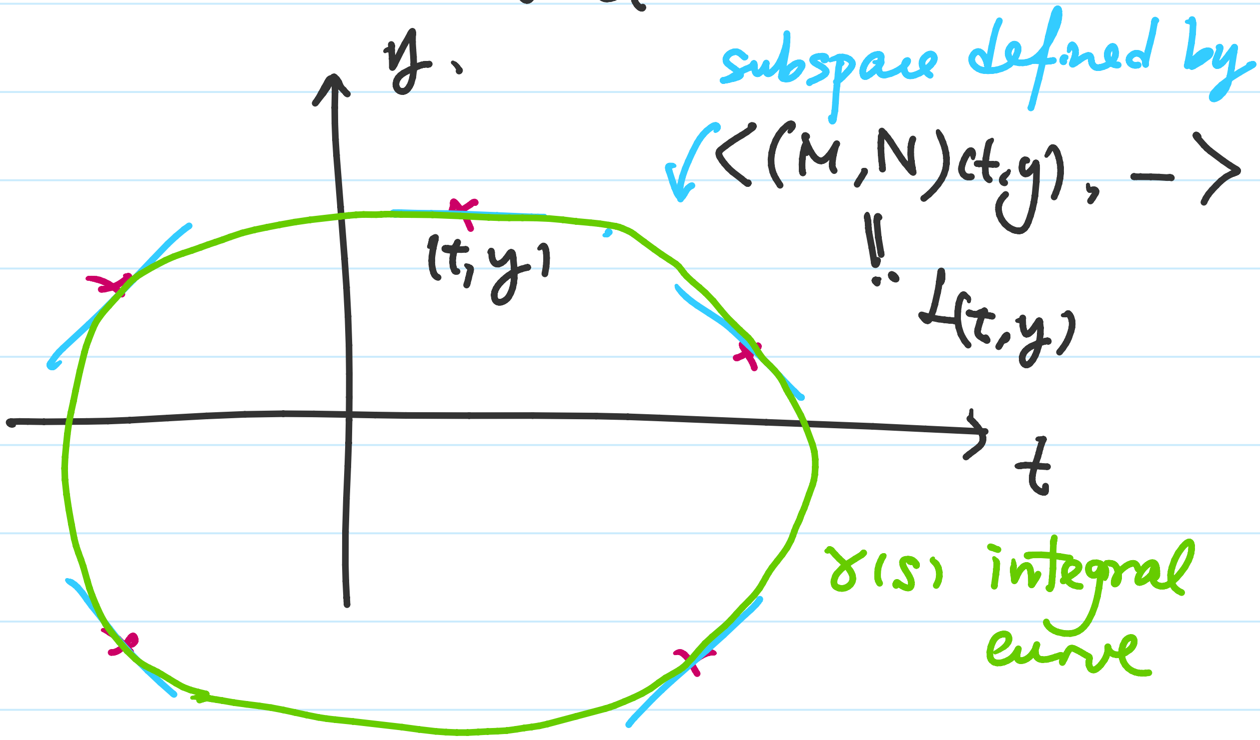
Example: $M(t,y) = y\cos(ty)$, $N(t,y) = t\cos(ty)$.

We can choose $\psi = \sin(ty) + a \in \mathbb{R}$.

General solution: $\sin(ty(t)) = c.$

Geometric viewpoint for exact equation:

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0$$



Def: • A direction field is an association of 1-dimensional subspace $L_{(t,y)} \subseteq \mathbb{R}^2$ for each point (t,y).

$\gamma(s) = (\gamma_1(s), \gamma_2(s))$ is an integral curve

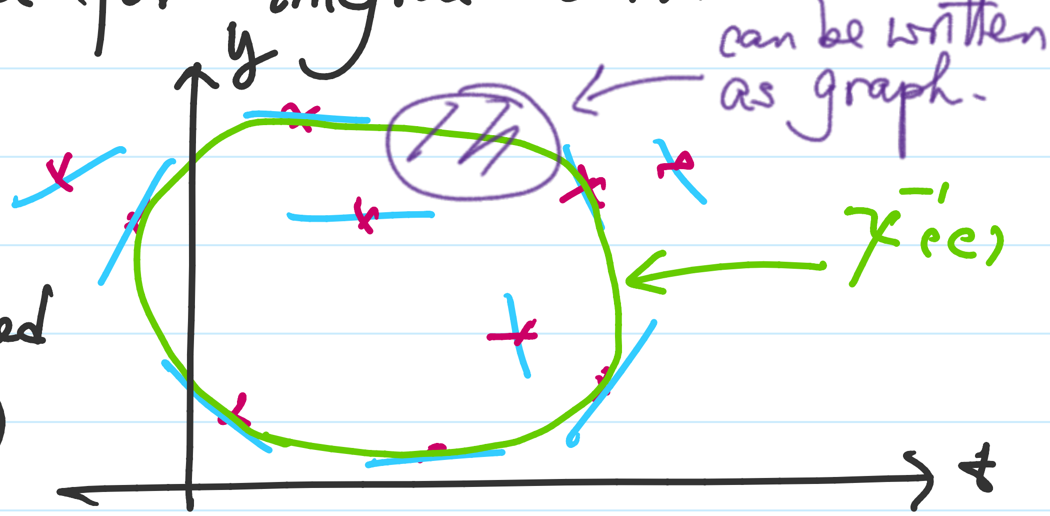
iff $\quad \gamma'(s) \in L_{\gamma(s)}$ or $M\gamma_1' + N\gamma_2' = 0$.

Prop: $y(t)$ solve $M(t,y)+N(t,y)\frac{dy}{dt}=0$

$\iff$ its graph $\gamma(t)=(t,y(t))$ is an integral curve.

Rk: Indeed we are solving for implicit formula for integral curve:

- Direction field defined by (M,N)



it is exact $\Longrightarrow \exists\, \chi$ s.t. $\frac{\partial \chi}{\partial t}=M, \frac{\partial \chi}{\partial y}=N$

Hence: The level set

$\chi^{-1}(c)$ is an implicit integral curve

We are actually solve for the level set!

- When $\chi^{-1}(c)$ can be parametrized as a graph $(t,y(t))$, it gives a sol.

Question: when is an equation $M(t,y) + N(t,y)\frac{dy}{dt}$ exact?

Assumption (*): $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial t}$ defined on $R = (a,b)\times(c,d)$ and continuous

Necessary condition: if $M = \frac{\partial \psi}{\partial t}$, $N = \frac{\partial \psi}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial t} = \frac{\partial^2 \psi}{\partial t \partial y} = \frac{\partial N}{\partial t}.$$

i.e. $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}}$

Thm: 1. $M(t,y)+N(t,y)\frac{dy}{dt}$ exact $\Longleftrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

under the assumption (*) above

Pf: $\Rightarrow$) $\because \frac{\partial^2 \psi}{\partial t \partial y} = \frac{\partial^2 \psi}{\partial y \partial t}$.

$\Leftarrow$) We want to construct such a ψ satisfying

$$\frac{\partial \psi}{\partial t} = M, \quad \frac{\partial \psi}{\partial y} = N.$$

Let $Q(t,y) = \int_{t_0}^{t} M(s,y)\,ds$

$\Rightarrow$ $\psi(t,y) = Q(t,y) + h(y)$

$\left(\because \frac{\partial}{\partial t}(\psi - Q) = 0 \Rightarrow \psi - Q \text{ function of } y\right)$

taking derivatives in y-direction

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds + \frac{dh}{dy}$$

$$N = \frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds + \frac{dh}{dy}$$

$$\frac{dh}{dy} = \underline{N - \frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds}$$

independent of t!

this should be independent of t.

$$\frac{\partial}{\partial t}\left(N - \frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds\right)$$

$$= \frac{\partial M}{\partial y} - \frac{\partial}{\partial t}\frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds$$

$$= \frac{\partial M}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial t}\int_{t_0}^{t} M(s,y)\,ds$$

$$= \frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} = 0.$$

$$\Rightarrow \quad h(y) = \int_{y_0}^{y} N(t_0, w)\,dw - \cancelto{0}{\int_{y_0}^{y}\frac{\partial}{\partial w}\int_{t_0}^{t_0} M(s,w)\,ds\,dw} + \cancelto{0}{e}$$

Not important

$$\boxed{\psi(t,y) = \int_{t_0}^{t} M(s,y)\,ds + \int_{y_0}^{y} N(t_0, w)\,dw}$$

$\blacksquare$

Rk: • This result depends heavily on the fact that $R = (a,b)\times(c,d)$ is <u>simply connected</u>

Example:

$$M(t,y) = y\cos(t) + 2te^{y}$$
$$N(t,y) = \sin(t) + t^2e^{y} - 1.$$

$$\frac{\partial M}{\partial y} = \cos(t) + 2te^{y}, \quad \frac{\partial N}{\partial t} = \cos(t) + 2te^{y}$$

it is exact, and we need to solve for ψ

$$\psi(t,y) = \int_{t_0}^{t} M(s,y)\,ds + \int_{y_0}^{y} N(t_0,w)\,dw$$

$$= \int_{t_0}^{t} (y\cos s + 2se^{y})\,ds + \int_{y_0}^{y} (\sin(t_0) + t_0^2e^{w} - 1)\,dw$$

$$= y\sin(s) + s^2e^{y}\Big|_{t_0}^{t} + (\sin(t_0)-1)w + t_0^2e^{w}\Big|_{y_0}^{y}$$

constant.

$$= y\sin(t) + t^2e^{y} + (y_0 - y) - y_0\sin(t_0) - t_0^2e^{y_0}$$

take $\psi = y\sin(t) + t^2e^{y} - y$

We have general implicity solution is given by

$$y(t)\sin(t) + t^2e^{y(t)} - y(t) = C.$$

Counter example:

$R = \mathbb{R}^2 \setminus \{0\}$, then this is NOT true

we let $M(t,y) = \dfrac{-y}{t^2+y^2}$, $N(t,y) = \dfrac{+t}{t^2+y^2}$

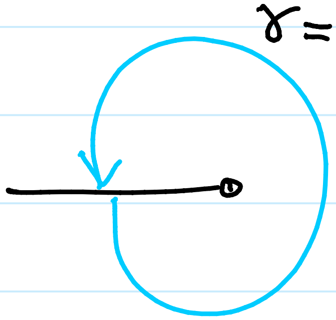
then we have

$$\frac{\partial M}{\partial y} = -\frac{t^2+y^2-2y^2}{t^2+y^2} = \frac{y^2-t^2}{t^2+y^2}$$

$$\frac{\partial N}{\partial t} = -\frac{t^2-y^2}{t^2+y^2} = \frac{\partial M}{\partial y}.$$

But $\nexists\ \psi$ s.t. $\dfrac{\partial \psi}{\partial t} = M$, $\dfrac{\partial \psi}{\partial y} = N$.

Reason: suppose we use polar coordinate (r, θ) and let $\gamma = (t(s), y(s))$



$\gamma(s)$, a counterclockwise circle

Check: $\dfrac{\partial \theta}{\partial t} = M$, $\dfrac{\partial \theta}{\partial y} = N$.

therefore: $\displaystyle\lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{2\pi-\varepsilon} \left(M \frac{dt}{ds} + N \frac{dy}{ds} \right) ds$

$$= \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{2\pi-\varepsilon} \big(\theta(t(s), y(s))\big)' \, ds = 2\pi$$

However, if we have such a ψ s.t.

$$\frac{\partial \psi}{\partial t} = M, \qquad \frac{\partial \psi}{\partial y} = N$$

$$\Rightarrow \int_0^{2\pi} \left(M \frac{dt}{ds} + N \frac{dy}{ds} \right) ds = \int_0^{2\pi} \left(\psi(t(s), y(s)) \right)' ds$$

$$= \psi(t(2\pi), y(2\pi)) - \psi(t(0), y(0)) = 0.$$

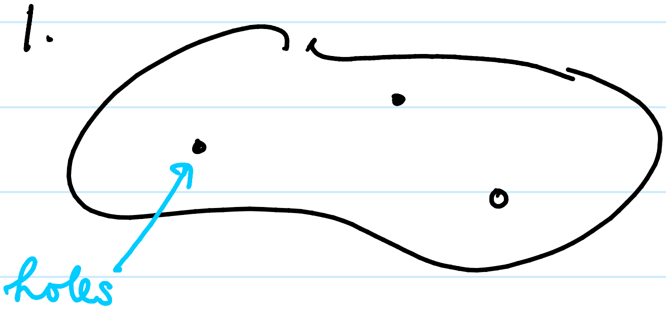
$$\left(\because \quad t(2\pi) = t(0), \quad y(2\pi) = y(0). \right)$$

Hence existence of such ψ leads to a contradiction

Roughly: simply connected domain $D \subseteq \mathbb{R}^2$ is those without "holes"

Ex: (non-simply connected)

1.



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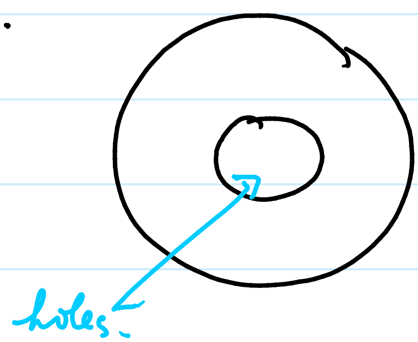


Extra story: $K = \{ (M, N) \mid M, N : D \to \mathbb{R}$ smooth fcns satisfying $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \}$

$\mathbb{R}$-vector space: $a(M_1, N_1) + (M_2, N_2) = (aM_1 + M_2, aN_1 + N_2)$

we let $\Omega^0 = \{ \psi \mid \psi : D \to \mathbb{R}$ smooth fcn$\}$

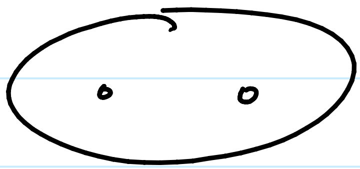
then we can define a linear map d

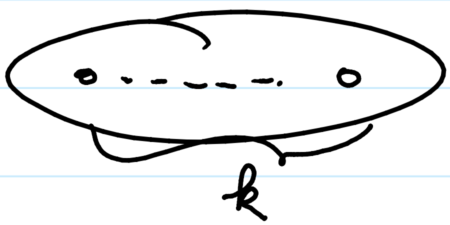
$$d : \Omega^0 \longrightarrow K$$

$$d(\psi) := \left(\underset{= M}{\frac{\partial \psi}{\partial x}} , \underset{= N}{\frac{\partial \psi}{\partial y}} \right)$$

then we have $H^1(D) := K \big/ d\Omega^0$ be the quotient vector space

<u>one can show:</u>

$D =$  $\Rightarrow \dim(H^1(D)) = 2.$

$D =$ 

$\Rightarrow \dim(H^1(D)) = k.$

in the order words, it is an invariant detect # of holes in D